DAY FIVE

Matrices

Learning & Revision for the Day

- Matrix
- Types of Matrices
- Equality of Matrices
- Algebra of Matrices
- Transpose of a Matrix
- Some Special Matrices
- Trace of a Matrix
- Equivalent Matrices
- Invertible Matrices

Matrix

- A **matrix** is an arrangement of numbers in rows and columns.
- A matrix having m rows and n columns is called a matrix of order $m \times n$ and the number of elements in this matrix will be mn.
- A matrix of order $m \times n$ is of the form $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$

Some important terms related to matrices

- The element in the ith row and jth column is denoted by a_{ij}.
- The elements $a_{11}, a_{22}, a_{33}, \dots\dots$ are called diagonal elements.
- The line along which the diagonal elements lie is called the principal diagonal or simply the diagonal of the matrix.

Types of Matrices

- If all elements of a matrix are zero, then it is called a **null** or **zero matrix** and it is denoted by O.
- A matrix which has only one row and any number of columns is called a **row matrix** and if it has only one column and any number of rows, then it is called a **column matrix**.
- If in a matrix, the number of rows and columns are equal, then it is called a **square matrix**. If $A = [a_{ij}]_{n \times n}$, then it is known as square matrix of order n.
- If in a matrix, the number of rows is less/greater than the number of columns, then it is called **rectangular matrix**.
- If in a square matrix, all the non-diagonal elements are zero, it is called a **diagonal matrix**.

- If in a square matrix, all non-diagonal elements are zero and diagonal elements are equal, then it is called a **scalar matrix**.
- If in a square matrix, all non-diagonal elements are zero and diagonal elements are unity, then it is called an **unit** (identity) **matrix**. We denote the identity matrix of order n by I_n and when order is clear from context then we simply write it as I.
- In a square matrix, if $a_{ij} = 0, \forall\ i > j$, then it is called an **upper triangular matrix** and if $a_{ij} = 0, \forall i < j$, then it is called a **lower triangular matrix**.

NOTE
- The diagonal elements of diagonal matrix may or may not be zero.

Equality of Matrices

Two matrices A and B are said to be equal, if they are of same order and all the corresponding elements are equal.

Algebra of Matrices

- If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two matrices of same order, then $A + B = [a_{ij} + b_{ij}]_{m \times n}$ and $A - B = [a_{ij} - b_{ij}]_{m \times n}$, where $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$.
- If $A = [a_{ij}]$ be an $m \times n$ matrix and k be any scalar, then, $kA = [ka_{ij}]_{m \times n}$.
- If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ be any two matrices such that number of columns of A is equal to the number of rows of B, then the product matrix $AB = [c_{ij}]$, of order $m \times p$, where $c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$.

Some Important Properties

- $A + B = B + A$ (Commutativity of addition)
- $(A + B) + C = A + (B + C)$ (Associativity of addition)
- $\alpha\,(A + B) = \alpha A + \alpha B$, where α is any scalar.
- $(\alpha + \beta)\,A = \alpha A + \beta A$, where α and β are any scalars.
- $\alpha\,(\beta A) = (\alpha\beta)\,A$, where α and β are any scalars.
- $(AB)C = A\,(BC)$ (Associativity of multiplication)
- $AI = A = IA$
- $A\,(B + C) = AB + AC$ (Distributive property)

NOTE
- $A^2 = A \cdot A,\ A^3 = A \cdot A \cdot A = A^2 \cdot A^1, \ldots$
- If the product AB is possible, then it is not necessary that the product BA is also possible. Also, it is not necessary that $AB = BA$.
- The product of two non-zero matrices can be a zero matrix.

Transpose of a Matrix

Let A be $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A and is denoted by A' or A^C or A^T.

If A be $m \times n$ matrix, A' will be $n \times m$ matrix.

Important Results

(i) If A and B are two matrices of order $m \times n$, then $(A \pm B)' = A' \pm B'$

(ii) If k is a scalar, then $(k\,A)' = k\,A'$

(iii) $(A')' = A$

(iv) $(AB)' = B'\,A'$

(v) $(A^n)' = (A')^n$

Some Special Matrices

- A square matrix A is called an **idempotent matrix**, if it satisfies the relation $A^2 = A$.
- A square matrix A is called **nilpotent matrix** of order k, if it satisfies the relation $A^k = O$, for some $k \in N$.
- The least value of k is called the index of the nilpotent matrix A.
- A square matrix A is called an **involutory matrix**, if it satisfies the relation $A^2 = I$.
- A square matrix A is called an **orthogonal matrix**, if it satisfies the relation $AA' = I$ or $A'\,A = I$.
- A square matrix A is called **symmetric matrix**, if it satisfies the relation $A' = A$.
- A square matrix A is called **skew-symmetric matrix**, if it satisfies the relation $A' = -A$.

NOTE
- If A and B are idempotent matrices, then $A + B$ is idempotent iff $AB = -BA$.
- If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ is orthogonal, then

  $\Sigma a_i^2 = \Sigma b_i^2 = \Sigma c_i^2 = 1$ and $\Sigma a_i b_i = \Sigma b_i c_i = \Sigma a_i c_i = 0$
- If A and B are symmetric matrices of the same order, then
  (i) AB is symmetric if and only if $AB = BA$.
  (ii) $A \pm B$, $AB + BA$ are also symmetric matrices.
- If A and B are two skew-symmetric matrices, then
  (i) $A \pm B$, $AB - BA$ are skew-symmetric matrices.
  (ii) $AB + BA$ is a symmetric matrix.
- Every square matrix can be uniquely expressed as the sum of symmetric and skew-symmetric matrices.

  i.e. $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$, where $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$ are symmetric and skew-symmetric respectively.

Trace of a Matrix

The sum of the diagonal elements of a square matrix A is called the trace of A and is denoted by $\text{tr}(A)$.

(i) $\text{tr}(\lambda A) = \lambda\, \text{tr}(A)$ (ii) $\text{tr}(A) = \text{tr}(A')$

(iii) $\text{tr}(AB) = \text{tr}(BA)$

Equivalent Matrices

Two matrices A and B are said to be **equivalent**, if one is obtained from the other by one or more elementary operations and we write $A \sim B$.

Following types of operations are called ***elementary operations***.

(i) Interchanging any two rows (columns).
This transformation is indicated by
$$R_i \leftrightarrow R_j\ (C_i \leftrightarrow C_j)$$

(ii) Multiplication of the elements of any row (column) by a non-zero scalar quantity, indicated as
$$R_i \to kR_i\ (C_i \to kC_i)$$

(iii) Addition of constant multiple of the elements of any row (column) to the corresponding elements of any other row (column), indicated as
$$R_i \to R_i + kR_j\ (C_i \to C_i + kC_j).$$

Invertible Matrices

- A square matrix A of order n is said to be **invertible** if there exists another square matrix B of order n such that $AB = BA = I$.
- The matrix B is called the inverse of matrix A and it is denoted by A^{-1}.

Some Important Results

- Inverse of a square matrix, if it exists, is unique.
- $AA^{-1} = I = A^{-1}A$
- If A and B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^{-1})^T = (A^T)^{-1}$
- If A is symmetric, then A^{-1} will also be symmetric matrix.
- Every orthogonal matrix is invertible.

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 If $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then which of the following is correct? ➜ **NCERT Exemplar**

(a) $(A+B)\cdot(A-B) = A^2 + B^2$ (b) $(A+B)\cdot(A-B) = A^2 - B^2$
(c) $(A+B)\cdot(A-B) = I$ (d) None of these

2 If p, q, r are 3 real numbers satisfying the matrix equation, $[p\ q\ r]\begin{bmatrix} 3 & 4 & 1 \\ 3 & 2 & 3 \\ 2 & 0 & 2 \end{bmatrix} = [3\ \ 0\ \ 1]$, then $2p + q - r$ is equal to ➜ **JEE Mains 2013**

(a) -3 (b) -1 (c) 4 (d) 2

3 In a upper triangular matrix $n \times n$, minimum number of zeroes is

(a) $\dfrac{n(n-1)}{2}$ (b) $\dfrac{n(n+1)}{2}$
(c) $\dfrac{2n(n-1)}{2}$ (d) None of these

4 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$; $a, b \in N$. Then,

(a) there exists more than one but finite number of B's such that $AB = BA$
(b) there exists exactly one B such that $AB = BA$
(c) there exist infinitely many B's such that $AB = BA$
(d) there cannot exist any B such that $AB = BA$

5 If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, then

(a) AB, BA exist and are equal
(b) AB, BA exist and are not equal
(c) AB exists and BA does not exist
(d) AB does not exist and BA exists

6 If $\omega \neq 1$ is the complex cube root of unity and matrix $H = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$, then H^{70} is equal to

(a) H (b) 0 (c) $-H$ (d) H^2

7 If A and B are 3×3 matrices such that $AB = A$ and $BA = B$, then

(a) $A^2 = A$ and $B^2 \neq B$ (b) $A^2 \neq A$ and $B^2 = B$
(c) $A^2 = A$ and $B^2 = B$ (d) $A^2 \neq A$ and $B^2 \neq B$

8 For each real number x such that $-1 < x < 1$, let $A(x) = \begin{bmatrix} \dfrac{1}{1-x} & \dfrac{-x}{1-x} \\ \dfrac{-x}{1-x} & \dfrac{1}{1-x} \end{bmatrix}$ and $z = \dfrac{x+y}{1+xy}$. Then,

(a) $A(z) = A(x) + A(y)$
(b) $A(z) = A(x)\,[A(y)]^{-1}$
(c) $A(z) = A(x)\cdot A(y)$
(d) $A(z) = A(x) - A(y)$

9 If $A(\alpha)=\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$, then $A(\alpha)\,A(\beta)$ is equal to

(a) $A(\alpha\beta)$ (b) $A(\alpha+\beta)$ (c) $A(\alpha-\beta)$ (d) None

10 If A is 3×4 matrix and B is a matrix such that $A'B$ and BA' are both defined, then B is of the type

(a) 4×3 (b) 3×4 (c) 3×3 (d) 4×4

11 If $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ a & 2 & b\end{bmatrix}$ is a matrix satisfying the equation $AA^T=9I$, where I is 3×3 identity matrix, then the ordered pair (a,b) is equal to → **JEE Mains 2015**

(a) $(2,-1)$ (b) $(-2,1)$ (c) $(2,1)$ (d) $(-2,-1)$

12 If $E=\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$ and $F=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$, then E^2F+F^2E

(a) F (b) E (c) 0 (d) None

13 If A and B are two invertible matrices and both are symmetric and commute each other, then

(a) both $A^{-1}B$ and $A^{-1}B^{-1}$ are symmetric
(b) neither $A^{-1}B$ nor $A^{-1}B^{-1}$ are symmetric
(c) $A^{-1}B$ is symmetric but $A^{-1}B^{-1}$ is not symmetric
(d) $A^{-1}B^{-1}$ is symmetric but $A^{-1}B$ is not symmetric

14 If neither α nor β are multiples of $\pi/2$ and the product AB of matrices

$$A=\begin{bmatrix}\cos^2\alpha & \sin\alpha\cos\alpha\\ \cos\alpha\sin\alpha & \sin^2\alpha\end{bmatrix}$$

and $$B=\begin{bmatrix}\cos^2\beta & \cos\beta\sin\beta\\ \cos\beta\sin\beta & \sin^2\beta\end{bmatrix}$$

is null matrix, then $\alpha-\beta$ is

(a) 0 (b) multiple of π
(c) an odd multiple of $\pi/2$ (d) None of these

15 The matrix $\begin{bmatrix}1 & 2 & 3\\ 1 & 2 & 3\\ -1 & -2 & -3\end{bmatrix}$ is

(a) idempotent (b) nilpotent
(c) involutary (d) orthogonal

16 If $A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$, then

(a) A is skew-symmetric (b) symmetric
(c) idempotent (d) orthogonal

17 If $A=\begin{bmatrix}a & a^2-1 & -2\\ a+1 & 1 & a^2+4\\ -2 & 4a & 5\end{bmatrix}$ is symmetric, then a is

(a) -2 (b) 2 (c) -1 (d) None

18 If $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ x & 2 & y\end{bmatrix}$ and $A^TA=AA^T=I$, then xy is equal to

(a) -1 (b) 1 (c) 2 (d) -2

19 If A and B are symmetric matrices of the same order and $X=AB+BA$ and $Y=AB-BA$, then $(XY)^T$ is equal to

(a) XY (b) YX
(c) $-YX$ (d) None of these

20 Let $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & -2 & 4\end{bmatrix}$, $I=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ and $A^{-1}=\left[\frac{1}{6}(A^2+cA+dI)\right]$. The values of c and d are

(a) $(-6,-11)$ (b) $(6,11)$
(c) $(-6,11)$ (d) $(6,-11)$

21 Elements of a matrix A of order 9×9 are defined as $a_{ij}=\omega^{i+j}$ (where ω is cube root of unity), then trace (A) of the matrix is

(a) 0 (b) 1 (c) ω (d) ω^2

22 If $A=\begin{bmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{bmatrix}$ and $A^{-1}=\frac{1}{10}\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & \alpha\\ 1 & -2 & 3\end{bmatrix}$, then α is equal to

(a) -2 (b) 5 (c) 2 (d) -1

23 If A is skew-symmetric and $B=(I-A)^{-1}(I+A)$, then B is

(a) symmetric
(b) skew-symmetric
(c) orthogonal
(d) None of the above

24 Let A be a square matrix satisfying $A^2+5A+5I=O$. The inverse of $A+2I$ is equal to

(a) $A-2I$ (b) $A+3I$
(c) $A-3I$ (d) does not exist

25 Let $A=\begin{bmatrix}1 & 0\\ 1/3 & 1\end{bmatrix}$. Then A^{48} is

(a) $\begin{bmatrix}1 & 0\\ (1/3)^{48} & 1\end{bmatrix}$ (b) $\begin{bmatrix}1 & 0\\ \frac{3}{2}\left[1-\frac{1}{3^{48}}\right] & 1\end{bmatrix}$

(c) $\begin{bmatrix}1 & 0\\ 16 & 1\end{bmatrix}$ (d) None of these

26 If X is any matrix of order $n\times p$ and I is an identity matrix of order $n\times n$, then the matrix $M=I-X(X'X)^{-1}X'$ is

I. Idempotent matrix
II. $MX=O$

(a) Only I is correct (b) Only II is correct
(c) Both I and II are correct (d) None of them is correct

27 Let A and B be two symmetric matrices of order 3.

Statement I $A(BA)$ and $(AB)A$ are symmetric matrices.

Statement II AB is symmetric matrix, if matrix multiplication of A with B is commutative.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

28 Consider the following relation R on the set of real square matrices of order 3.

$R = \{(A, B) : A = P^{-1}BP$ for some invertible matrix $P\}$

Statement I R is an equivalence relation.

Statement II For any two invertible 3×3 matrices M and $N, (MN)^{-1} = N^{-1}M^{-1}$.

(a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true; Statement II is correct explanation of Statement I
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation of Statement I
(d) Statement I is true, Statement II is false

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 If $A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$, then $I + 2A + 3A^2 + \dots \infty$ is equal to

(a) $\begin{bmatrix} 4 & 1 \\ -4 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 5 & 2 \\ -8 & -3 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 2 \\ -3 & -8 \end{bmatrix}$

2 The matrix A that commute with the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is

(a) $A = \frac{1}{2}\begin{pmatrix} 2a & 2b \\ 3b & 2a + 3b \end{pmatrix}$ (b) $A = \frac{1}{2}\begin{pmatrix} 2b & 2a \\ 3a & 2a + 3b \end{pmatrix}$
(c) $A = \frac{1}{3}\begin{pmatrix} 2a + 3b & 2a \\ 3a & 2a + 3b \end{pmatrix}$ (d) None of these

3 The total number of matrices that can be formed using 5 different letters such that no letter is repeated in any matrix, is

(a) 5! (b) 2×5^5
(c) $2 \times (5!)$ (d) None of these

4 If A is symmetric and B is a skew-symmetric matrix, then for $n \in N$, which of the following is not correct?

(a) A^n is symmetric
(b) B^n is symmetric if n is even
(c) A^n is symmetric if n is odd only
(d) B^n is skew-symmetric if n is odd

5 Consider three matrices $X = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}, Y = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}$ and $Z = \begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix}$. Then, the value of the sum

$tr(X) + tr\left(\frac{XYZ}{2}\right) + tr\left(\frac{X(YZ)^2}{4}\right) + tr\left(\frac{X(YZ)^3}{8}\right) + \dots$ to ∞ is

(a) 6 (b) 9
(c) 12 (d) None of these

6 If both $A - \frac{1}{2}I$ and $A + \frac{1}{2}I$ are orthogonal matrices, then

(a) A is orthogonal
(b) A is skew-symmetric matrix
(c) A is symmetric matrix
(d) None of the above

7 If $A = \begin{bmatrix} \frac{-1 + i\sqrt{3}}{2i} & \frac{-1 - i\sqrt{3}}{2i} \\ \frac{1 + i\sqrt{3}}{2i} & \frac{1 - i\sqrt{3}}{2i} \end{bmatrix}$, $i = \sqrt{-1}$ and $f(x) = x^2 + 2$, then $f(A)$ is equal to

(a) $\left(\frac{5 - i\sqrt{3}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\left(\frac{3 - i\sqrt{3}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $(2 + i\sqrt{3})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

8 If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, then A^n is equal to

(a) $2^{n-1}A - (n-1)I$ (b) $nA - (n-1)I$
(c) $2^{n-1}A + (n-1)I$ (d) $nA + (n-1)I$

9 Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Let $A^n = [b_{ij}]_{2\times2}$. Define $\lim_{n\to\infty} A^n = \lim_{n\to\infty} [b_{ij}]_{2\times2}$. Then $\lim_{n\to\infty}\left(\frac{A^n}{n}\right)$ is

(a) zero matrix (b) unit matrix
(c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (d) limit does not exist

10 If B is skew-symmetric matrix of order n and A is $n \times 1$ column matrix and $A^T BA = [p]$, then

(a) $p < 0$ (b) $p = 0$
(c) $p > 0$ (d) Nothing can be said

11 If A, B and $A+B$ are idempotent matrices, then AB is equal to

(a) BA (b) $-BA$ (c) I (d) O

12 If $P=\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q=PAP^T$, then $P^TQ^{2019}P$ is equal to

(a) $\begin{bmatrix} 1 & 2019 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 4+2019\sqrt{3} & 6057 \\ 2019 & 4-2019\sqrt{3} \end{bmatrix}$

(c) $\frac{1}{4}\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix}$

(d) $\frac{1}{4}\begin{bmatrix} 2019 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2019 \end{bmatrix}$

13 Which of the following is an orthogonal matrix?

(a) $\frac{1}{7}\begin{bmatrix} 6 & 2 & -3 \\ 2 & 3 & 6 \\ 3 & -6 & 2 \end{bmatrix}$ (b) $\frac{1}{7}\begin{bmatrix} 6 & 2 & 3 \\ 2 & -3 & 6 \\ 3 & 6 & -2 \end{bmatrix}$

(c) $\frac{1}{7}\begin{bmatrix} -6 & -2 & -3 \\ 2 & 3 & 6 \\ -3 & 6 & 2 \end{bmatrix}$ (d) $\frac{1}{7}\begin{bmatrix} 6 & -2 & 3 \\ 2 & 2 & -3 \\ -6 & 2 & 3 \end{bmatrix}$

14 If $A_1, A_3, \dots, A_{2n-1}$ are n skew-symmetric matrices of same order, then $B=\sum_{r=1}^{n}(2r-1)(A_{2r-1})^{2r-1}$ will be

(a) symmetric
(b) skew-symmetric
(c) neither symmetric nor skew-symmetric
(d) data not adequate

15 Let matrix $A=\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, where a, b, c are real positive numbers with $abc=1$. If $A^T A=I$, then $a^3+b^3+c^3$ is

(a) 3 (b) 4
(c) 2 (d) None of these

16 If A is an 3×3 non-singular matrix such that $AA'=A'A$ and $B=A^{-1}A'$, then BB' equals **→ JEE Mains 2014**

(a) $(B^{-1})'$ (b) $I+B$
(c) I (d) B^{-1}

17 A is a 3×3 matrix with entries from the set $\{-1, 0, 1\}$. The probability that A is neither symmetric nor skew-symmetric is

(a) $\frac{3^9-3^6-3^3+1}{3^9}$ (b) $\frac{3^9-3^6-3^3}{3^9}$

(c) $\frac{3^9-3^6+1}{3^9}$ (d) $\frac{3^9-3^3+1}{3^9}$

ANSWERS

SESSION 1	**1.** (d)	**2.** (a)	**3.** (a)	**4.** (c)	**5.** (b)	**6.** (a)	**7.** (c)	**8.** (c)	**9.** (b)	**10.** (b)
	11. (d)	**12.** (b)	**13.** (a)	**14.** (c)	**15.** (b)	**16.** (d)	**17.** (b)	**18.** (c)	**19.** (c)	**20.** (c)
	21. (a)	**22.** (b)	**23.** (c)	**24.** (b)	**25.** (c)	**26.** (c)	**27.** (b)	**28.** (c)		
SESSION 2	**1.** (c)	**2.** (a)	**3.** (c)	**4.** (c)	**5.** (a)	**6.** (b)	**7.** (d)	**8.** (b)	**9.** (a)	**10.** (b)
	11. (b)	**12.** (a)	**13.** (a)	**14.** (b)	**15.** (d)	**16.** (c)	**17.** (a)			

Hints and Explanations

SESSION 1

1 Here,

$$A+B=\begin{bmatrix}0&1\\1&1\end{bmatrix}+\begin{bmatrix}0&-1\\1&0\end{bmatrix}=\begin{bmatrix}0&0\\2&1\end{bmatrix}$$

$$A-B=\begin{bmatrix}0&1\\1&1\end{bmatrix}-\begin{bmatrix}0&-1\\1&0\end{bmatrix}=\begin{bmatrix}0&2\\0&1\end{bmatrix}$$

$$A^2=A\cdot A=\begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}0&1\\1&1\end{bmatrix}$$

$$=\begin{bmatrix}0+1&0+1\\0+1&1+1\end{bmatrix}=\begin{bmatrix}1&1\\1&2\end{bmatrix}$$

and $B^2=B\cdot B=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\cdot\begin{bmatrix}0&-1\\1&0\end{bmatrix}$

$$=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$$

$$\therefore\ A^2+B^2=\begin{bmatrix}1&1\\1&2\end{bmatrix}+\begin{bmatrix}-1&0\\0&-1\end{bmatrix}=\begin{bmatrix}0&1\\1&1\end{bmatrix}$$

$$A^2-B^2=\begin{bmatrix}1&1\\1&2\end{bmatrix}-\begin{bmatrix}-1&0\\0&-1\end{bmatrix}=\begin{bmatrix}2&1\\1&3\end{bmatrix}$$

and $(A+B)(A-B)=\begin{bmatrix}0&0\\2&1\end{bmatrix}\begin{bmatrix}0&2\\0&1\end{bmatrix}$

$$=\begin{bmatrix}0+0&0+0\\0+0&4+1\end{bmatrix}=\begin{bmatrix}0&0\\0&5\end{bmatrix}$$

Clearly, $(A+B)(A-B)\neq A^2-B^2$

$$\neq A^2+B^2\neq I.$$

2 $[3p+3q+2r,\ 4p+2q+0,\ p+3q+2r]=[3\ 0\ 1]$

$\Rightarrow 3p+3q+2r=3,\ 4p+2q=0,\ p+3q+2r=1$

$\Rightarrow p=1, q=-2, r=3$

$\therefore\ 2p+q-r=2-2-3=-3$

3 We know that, a square matrix $A=[a_{ij}]$ is said to be an upper triangular matrix if $a_{ij}=0,\ \forall\ i>j$.

Consider, an upper triangular matrix

$$A=\begin{bmatrix}1&2&3\\0&5&6\\0&0&7\end{bmatrix}_{3\times3}$$

Here, number of zeroes $=3=\dfrac{3(3-1)}{2}$

$\therefore$ Minimum number of zeroes

$$=\frac{n(n-1)}{2}$$

4 Clearly, $AB=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}a&0\\0&b\end{bmatrix}=\begin{bmatrix}a&2b\\3a&4b\end{bmatrix}$

and $BA=\begin{bmatrix}a&0\\0&b\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}a&2a\\3b&4b\end{bmatrix}$

If $AB=BA$, then $a=b$.

Hence, $AB=BA$ is possible for infinitely many values of B's.

5 Here, A is 2×3 matrix and B is 3×2 matrix.

$\therefore$ Both AB and BA exist, and AB is a 2×2 matrix, while BA is 3×3 matrix.

$\therefore\qquad AB\neq BA.$

6 Clearly,

$$H^2=\begin{bmatrix}\omega&0\\0&\omega\end{bmatrix}\begin{bmatrix}\omega&0\\0&\omega\end{bmatrix}=\begin{bmatrix}\omega^2&0\\0&\omega^2\end{bmatrix}$$

$$H^3=\begin{bmatrix}\omega^2&0\\0&\omega^2\end{bmatrix}\begin{bmatrix}\omega&0\\0&\omega\end{bmatrix}=\begin{bmatrix}\omega^3&0\\0&\omega^3\end{bmatrix}$$

$$\therefore\ H^{70}=\begin{bmatrix}\omega^{70}&0\\0&\omega^{70}\end{bmatrix}=\begin{bmatrix}\omega^{69}\cdot\omega&0\\0&\omega^{69}\cdot\omega\end{bmatrix}$$

$$=\begin{bmatrix}(\omega^3)^{23}\cdot\omega&0\\0&(\omega^3)^{23}\cdot\omega\end{bmatrix}$$

$$=\begin{bmatrix}\omega&0\\0&\omega\end{bmatrix}=H\qquad[\because\omega^3=1]$$

7 Since, $AB=A$

$\therefore\qquad B=I\ \Rightarrow\ B^2=B$

Similarly, $BA=B$

$\Rightarrow\qquad A=I$

$\Rightarrow\qquad A^2=A$

Hence, $A^2=A$ and $B^2=B$

8 We have,

$$A(x)=\frac{1}{1-x}\begin{bmatrix}1&-x\\-x&1\end{bmatrix}\qquad\ldots\text{(i)}$$

$$\therefore\quad A(y)=\frac{1}{1-y}\begin{bmatrix}1&-y\\-y&1\end{bmatrix}\qquad\ldots\text{(ii)}$$

and $A(z)=\dfrac{1}{1-\dfrac{(x+y)}{1+xy}}\begin{bmatrix}1&-\dfrac{(x+y)}{1+xy}\\-\dfrac{(x+y)}{1+xy}&1\end{bmatrix}$

$$=\frac{1+xy}{1+xy-x-y}\begin{bmatrix}1&-\dfrac{(x+y)}{1+xy}\\-\dfrac{(x+y)}{1+xy}&1\end{bmatrix}$$

$$=\frac{1+xy}{(1-x)(1-y)}\begin{bmatrix}1&-\dfrac{(x+y)}{1+xy}\\-\dfrac{(x+y)}{1+xy}&1\end{bmatrix}$$

$$=\frac{1}{(1-x)(1-y)}\begin{bmatrix}1+xy&-(x+y)\\-(x+y)&1+xy\end{bmatrix}\qquad\ldots\text{(iii)}$$

Now, consider

$$A(x)\cdot A(y)=\frac{1}{(1-x)(1-y)}\cdot\begin{bmatrix}1&-x\\-x&1\end{bmatrix}\begin{bmatrix}1&-y\\-y&1\end{bmatrix}$$

$$=\frac{1}{(1-x)(1-y)}\cdot\begin{bmatrix}1+xy&-(x+y)\\-(x+y)&1+xy\end{bmatrix}\qquad\ldots\text{(iv)}$$

From Eqs. (iii) and (iv), we get

$$A(z)=A(x)\cdot A(y).$$

9 $A(\alpha)\,A(\beta)=\begin{bmatrix}\cos\alpha&-\sin\alpha&0\\\sin\alpha&\cos\alpha&0\\0&0&1\end{bmatrix}\times\begin{bmatrix}\cos\beta&-\sin\beta&0\\\sin\beta&\cos\beta&0\\0&0&1\end{bmatrix}$

$$=\begin{bmatrix}\cos(\alpha+\beta)&-\sin(\alpha+\beta)&0\\\sin(\alpha+\beta)&\cos(\alpha+\beta)&0\\0&0&1\end{bmatrix}$$

$$=A(\alpha+\beta)$$

10 Clearly, order of A' is 4×3.

Now, for $A'B$ to be defined, order of B should be $3\times m$ and for BA' to be defined, order of B should be $n\times4$.

Thus, for both $A'B$ and BA' to be defined, order of B should be 3×4.

11 Given, $A=\begin{bmatrix}1&2&2\\2&1&-2\\a&2&b\end{bmatrix}$

$$\Rightarrow\quad A^T=\begin{bmatrix}1&2&a\\2&1&2\\2&-2&b\end{bmatrix}$$

Now, $AA^T=\begin{bmatrix}1&2&2\\2&1&-2\\a&2&b\end{bmatrix}\begin{bmatrix}1&2&a\\2&1&2\\2&-2&b\end{bmatrix}$

$$=\begin{bmatrix}9&0&a+4+2b\\0&9&2a+2-2b\\a+4+2b&2a+2-2b&a^2+4+b^2\end{bmatrix}$$

It is given that, $AA^T=9I$

$$\Rightarrow\begin{bmatrix}9&0&a+4+2b\\0&9&2a+2-2b\\a+4+2b&2a+2-2b&a^2+4+b^2\end{bmatrix}=\begin{bmatrix}9&0&0\\0&9&0\\0&0&9\end{bmatrix}$$

On comparing, we get

$$a+4+2b=0$$

$$\Rightarrow\qquad a+2b=-4\qquad\ldots\text{(i)}$$

$$2a+2-2b=0$$

$\Rightarrow \quad a-b=-1 \quad \ldots\text{(ii)}$

and $\quad a^2+4+b^2=9 \quad \ldots\text{(iii)}$

On solving Eqs. (i) and (ii), we get

$$a=-2, b=-1$$

This satisfies Eq. (iii) also.

Hence, $(a,b) \equiv (-2,-1)$

12 F is unit matrix $\Rightarrow F^2 = F$

and $\quad E^2F + F^2E = E^2 + E$

Also, $E^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore E^2 + E = E.$

13 Consider, $(A^{-1}B)^T = B^T (A^{-1})^T$

$= B^T(A^T)^{-1} = B\,A^{-1}$

$[\because A^T = A \text{ and } B^T = B]$

$= A^{-1}B$

$[\because AB = BA \Rightarrow A^{-1}(AB)A^{-1}$

$= A^{-1}(BA)\,A^{-1} \Rightarrow BA^{-1} = A^{-1}B]$

$\Rightarrow A^{-1}B$ is symmetric.

Now, consider

$(A^{-1}B^{-1})^T = ((BA)^{-1})^T$

$= ((AB)^{-1})^T \quad [\because AB = BA]$

$= (B^{-1}A^{-1})^T = (A^{-1})^T (B^{-1})^T$

$= (A^T)^{-1} (B^T)^{-1} = A^{-1}\,B^{-1}$

$\Rightarrow A^{-1}B^{-1}$ is also symmetric.

14 $AB = \begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix}$

$\times \begin{bmatrix} \cos^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & \sin^2\beta \end{bmatrix}$

$= \begin{bmatrix} \cos\alpha\cos\beta\cos(\alpha-\beta) & \cos\alpha\sin\beta\cos(\alpha-\beta) \\ \sin\alpha\cos\beta\cos(\alpha-\beta) & \sin\alpha\sin\beta\cos(\alpha-\beta) \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \quad \cos(\alpha-\beta) = 0$

$\Rightarrow \quad \alpha - \beta = (2n+1)\,\pi/2$

15 Let $\quad A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$

Then, $A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Hence, A is nilpotent matrix of index 2.

16 $A' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \neq A$ or $-A$.

$A\,A' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore A$ is orthogonal.

17 A is symmetric

$\Rightarrow a^2 - 1 = a + 1, a^2 + 4 = 4a$

$\Rightarrow a^2 - a - 2 = 0, a^2 - 4a + 4 = 0$

$\Rightarrow \quad a = 2.$

18 Since, A is orthogonal, each row is orthogonal to the other rows.

$\Rightarrow \quad R_1 \cdot R_3 = 0$

$\Rightarrow \quad x + 4 + 2y = 0$

Also, $\quad R_2 \cdot R_3 = 0$

$\Rightarrow \quad 2x + 2 - 2y = 0$

On solving, we get $x = -2,\ y = -1$

$\therefore \quad xy = 2$

19 Since, A and B are symmetric matrices

$\therefore \quad X = AB + BA$

will be a symmetric matrix and

$Y = AB - BA$ will be a skew-symmetric matrix.

Thus, we get $X^T = X$ and $Y^T = -Y$

Now, consider $(XY)^T = Y^TX^T$

$= (-Y)(X) = -YX$

20 Clearly, $6A^{-1} = A^2 + cA + dI$

$\Rightarrow (6A^{-1})A = (A^2 + cA + dI)A$

[$\because$ Post multiply both sides by A]

$\Rightarrow 6(A^{-1}A) = A^3 + cA^2 + dIA$

$\Rightarrow 6I = A^3 + cA^2 + dA$

$[\because A^{-1}A = I \text{ and } IA = A]$

$\Rightarrow A^3 + cA^2 + dA - 6I = O \quad \ldots\text{(i)}$

Here, $A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix}$

and $A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 19 \\ 0 & -38 & 46 \end{bmatrix}$

Now, from Eq. (i), we get

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 19 \\ 0 & -38 & 46 \end{bmatrix} + c\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix} + d\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 + c + d - 6 & 0 & 0 \\ 0 & -11 - c + d - 6 & 19 + 5c + d \\ 0 & -38 - 10c - 2d & 46 + 14c + 4d - 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \quad 1 + c + d - 6 = 0;$

$-11 - c + d - 6 = 0$

$\Rightarrow \quad c + d = 5;\ -c + d = 17$

On solving, we get $c = -6, d = 11.$

These value also satisfy other equations.

21 Clearly, $tr(A) = a_{11} + a_{22} + a_{33} + a_{44} + a_{55} + a_{66} + a_{77} + a_{88} + a_{99}$

$= \omega^2 + \omega^4 + \omega^6 + \omega^8 + \omega^{10} + \omega^{12} + \omega^{14} + \omega^{16} + \omega^{18}$

$= (\omega^2 + \omega + 1) + (\omega^2 + \omega + 1) + (\omega^2 + \omega + 1) [\because \omega^{3n} = 1, n \in N]$

$= 0 + 0 + 0 \quad [\because 1 + \omega + \omega^2 = 0]$

$= 0$

22 Clearly, $AA^{-1} = I$

Now, if R_1 of A is multiplied by C_3 of A^{-1}, we get $2 - \alpha + 3 = 0 \Rightarrow \alpha = 5$

23 Consider,

$BB^T = (I-A)^{-1}(I+A)(I+A)^T[(I-A)^{-1}]^T$

$= (I-A)^{-1}(I+A)(I-A)(I+A)^{-1}$

$= (I-A)^{-1}(I-A)(I+A)(I+A)^{-1}$

$= I \cdot I = I$

Hence, B is an orthogonal matrix.

24 We have, $A^2 + 5A + 5I = O$

$\Rightarrow \quad A^2 + 5A + 6I = I$

$\Rightarrow \quad (A + 2I)(A + 3I) = I$

$\Rightarrow \quad A + 2I$ and $A + 3I$ are inverse of each other.

25 If $A = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 1 & 0 \\ 2a & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 \\ 3a & 1 \end{bmatrix}, \ldots, A^n = \begin{bmatrix} 1 & 0 \\ na & 1 \end{bmatrix}$

Here, $a = 1/3,$

$\therefore \quad A^{48} = \begin{bmatrix} 1 & 0 \\ 16 & 1 \end{bmatrix}$

26 We have, $M = I - X(X'X)^{-1}X'$

$= I - X(X^{-1}(X')^{-1})X'$

$[\because (AB)^{-1} = B^{-1}A^{-1}]$

$= I - (XX^{-1})((X')^{-1}X')$

[by associative property]

$= I - I \times I \quad [\because AA^{-1} = I = A^{-1}A]$

$= I - I \quad [\because I^2 = I]$

$= O$

Clearly, $M^2 = O = M$

So, M is an idempotent matrix. Also, $MX = O$.

27 Given, $A^T = A$ and $B^T = B$

Statement I $[A(BA)]^T = (BA)^T \cdot A^T$

$$= (A^T B^T) A^T$$
$$= (AB)A = A(BA)$$

So, $A(BA)$ is symmetric matrix. Similarly, $(AB)A$ is symmetric matrix. Hence, Statement I is true. Also, Statement II is true but not a correct explanation of Statement I.

28 Given, $R = \{(A, B): A = P^{-1}BP$ for some invertible matrix $P\}$

For Statement I

(i) **Reflexive** ARA

$\Rightarrow \quad A = P^{-1}AP$

which is true only, if $P = I$. Thus, $A = P^{-1}AP$ for some invertible matrix P.

So, R is Reflexive.

(ii) **Symmetric**

$ARB \Rightarrow A = P^{-1}BP$

$\Rightarrow PAP^{-1} = P(P^{-1}BP)P^{-1}$

$\Rightarrow PAP^{-1} = (PP^{-1})B(PP^{-1})$

$\therefore \quad B = PAP^{-1}$

Now, let $Q = P^{-1}$

Then, $B = Q^{-1}AQ \Rightarrow BRA$

$\Rightarrow R$ is symmetric.

(iii) **Transitive** ARB and BRC

$\Rightarrow \quad A = P^{-1}BP$

and $B = Q^{-1}CQ$

$\Rightarrow \quad A = P^{-1}(Q^{-1}CQ)P$

$= (P^{-1}Q^{-1})C(QP)$

$= (QP)^{-1}C(QP)$

So, ARC.

$\Rightarrow R$ is transitive

So, R is an equivalence relation.

For Statement II It is always true that $(MN)^{-1} = N^{-1}M^{-1}$

Hence, both statements are true but second is not the correct explanation of first.

SESSION 2

1 Clearly, $A^2 = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$\therefore \; I + 2A + 3A^2 + \ldots = I + 2A$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -8 & -3 \end{bmatrix}$$

2 Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix that commute with $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Then,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix}$$

On equating the corresponding elements, we get

$a + 3b = a + 2c \Rightarrow 3b = 2c$ ...(i)

$2a + 4b = b + 2d \Rightarrow 2a + 3b = 2d$ ...(ii)

$c + 3d = 3a + 4c \Rightarrow a + c = d$ ...(iii)

$2c + 4d = 3b + 4d \Rightarrow 3b = 2c$ ...(iv)

Thus, A can be taken as

$$\begin{pmatrix} a & b \\ \frac{3b}{2} & a + \frac{3}{2}b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2a & 2b \\ 3b & 2a + 3b \end{pmatrix}$$

3 Clearly, matrix having five elements is of order 5×1 or 1×5.

$\therefore$ Total number of such matrices $= 2 \times 5!$.

4 $(A^n)' = (A\,A \cdots A)' = (A'A' \cdots A')$

$= (A')^n = A^n$ for all n

$\therefore A^n$ is symmetric for all $n \in N$.

Also, B is skew-symmetric

$\Rightarrow \quad B' = -B$.

$\therefore (B^n)' = (B\,B \cdots B)' = (B'B' \cdots B')$

$= (B')^n$

$= (-B)^n = (-1)^n B^n$.

$\Rightarrow B^n$ is symmetric if n is even and is skew-symmetric if n is odd.

5 Here, $YZ = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \; tr(X) + tr\left(\frac{XYZ}{2}\right) + tr\left(\frac{X(YZ)^2}{4}\right) + tr\left(\frac{X(YZ)^3}{8}\right) + \ldots$$

$$= tr(X) + tr\left(\frac{X}{2}\right) + tr\left(\frac{X}{4}\right) + \ldots$$

$$= tr(X) + \frac{1}{2}tr(X) + \frac{1}{4}tr(X) + \ldots$$

$$= tr(X)\left[1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\right]$$

$$= tr(X)\frac{1}{1 - \frac{1}{2}}$$

$$= 2\,tr(X) = 2(2 + 1) = 6$$

6 Since, both $A - \frac{1}{2}I$ and $A + \frac{1}{2}I$ are orthogonal, therefore, we have

$$\left(A - \frac{1}{2}I\right)'\left(A - \frac{1}{2}I\right) = I$$

$$\Rightarrow \left(A' - \frac{1}{2}I\right)\left(A - \frac{1}{2}I\right) = I \quad \ldots(i)$$

and $\left(A + \frac{1}{2}I\right)'\left(A + \frac{1}{2}I\right) = I$

$$\Rightarrow \left(A' + \frac{1}{2}I\right)\left(A + \frac{1}{2}I\right) = I \quad \ldots(ii)$$

From Eq. (i), we get

$$A'A - \frac{1}{2}IA' - \frac{1}{2}IA + \frac{1}{4}I = I$$

$$\Rightarrow A'A - \frac{1}{2}A' - \frac{1}{2}A + \frac{1}{4}I = I \quad \ldots(iii)$$

Similarly, from Eq. (ii), we get

$$A'A + \frac{1}{2}A' + \frac{1}{2}A + \frac{1}{4}I = I \quad \ldots(iv)$$

On subtracting Eq. (iii) from Eq. (iv), we get

$$A + A' = O$$

or $\quad A' = -A$

Hence, A is a skew-symmetric matrix.

7 We have, $A = \begin{bmatrix} \frac{\omega}{i} & \frac{\omega^2}{i} \\ -\frac{\omega^2}{i} & \frac{-\omega}{i} \end{bmatrix} = \frac{\omega}{i}\begin{bmatrix} 1 & \omega \\ -\omega & -1 \end{bmatrix}$

$$\therefore \quad A^2 = -\omega^2\begin{bmatrix} 1-\omega^2 & 0 \\ 0 & 1-\omega^2 \end{bmatrix}$$

$$= \begin{bmatrix} -\omega^2 + \omega^4 & 0 \\ 0 & -\omega^2 + \omega^4 \end{bmatrix}$$

$$= \begin{bmatrix} -\omega^2 + \omega & 0 \\ 0 & -\omega^2 + \omega \end{bmatrix}$$

$\because \quad f(x) = x^2 + 2$ [given]

$\therefore \quad f(A) = A^2 + 2I$

$$= \begin{bmatrix} -\omega^2 + \omega & 0 \\ 0 & -\omega^2 + \omega \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= (-\omega^2 + \omega + 2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= (3 + 2\omega)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= (2 + i\sqrt{3})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

8 $A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

....

....

$A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$

$$= \begin{bmatrix} n & 0 \\ n & n \end{bmatrix} - \begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix}$$

$$= nA - (n-1)I$$

9 $A^2 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$

Similarly, $A^3 = \begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix}$ etc

$\therefore \quad A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$

$= \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Now, $\lim_{n\to\infty} \frac{A^n}{n} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ as $\lim_{n\to\infty} \frac{b_{ij}}{n} = 0$

10 $A^T B A = [p] \Rightarrow (A^T BA)^T = [p]^T = [p]$

$\Rightarrow \quad A^T B^T A = A^T(-B) A = [p]$

$\Rightarrow \quad [-p] = [p] \Rightarrow p = 0.$

11 Since, A, B and $A + B$ are idempotent matrix

$\therefore A^2 = A; B^2 = B$ and $(A + B)^2 = A + B$

Now, consider $(A + B)^2 = A + B$

$\Rightarrow \quad A^2 + B^2 + AB + BA = A + B$

$\Rightarrow \quad A + B + AB + BA = A + B$

$\Rightarrow \quad AB = -BA$

12 P is orthogonal matrix as $P^T P = I$

$Q^{2019} = (PAP^T)(PAP^T)$

$\ldots(PAP^T) = PA^{2019}P^T$

$\therefore P^T Q^{2019} P = P^T \cdot PA^{2019}P^T \cdot P = A^{2019}$

Now, $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \quad A^{2019} = \begin{bmatrix} 1 & 2019 \\ 0 & 1 \end{bmatrix}$

13 We know that a matrix

$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ will be orthogonal if

$AA' = I$, which implies

$\Sigma a_i^2 = \Sigma b_i^2 = \Sigma c_i^2 = 1$

and $\quad \Sigma a_i b_i = \Sigma b_i c_i = \Sigma c_i a_i = 0$

Now, from the given options, only

$\frac{1}{7}\begin{bmatrix} 6 & 2 & -3 \\ 2 & 3 & 6 \\ 3 & -6 & 2 \end{bmatrix}$ satisfies these conditions.

Hence, $\frac{1}{7}\begin{bmatrix} 6 & 2 & -3 \\ 2 & 3 & 6 \\ 3 & -6 & 2 \end{bmatrix}$ is an orthogonal matrix.

14 We have,

$B = A_1 + 3A_3^3 + \ldots + (2n-1)A_{2n-1}^{2n-1}$

Now, $B^T = (A_1 + 3A_3^3 + \ldots + (2n-1)A_{2n-1}^{2n-1})^T$

$= A_1^T + (3A_3^3)^T + \ldots + ((2n-1)A_{2n-1}^{2n-1})^T$

$= A_1^T + 3(A_3^T)^3 + \ldots + (2n-1)(A_{2n-1}^T)^{2n-1}$

$= -A - 3A_3^3 - \ldots - (2n-1)A_{2n-1}^{2n-1}$

$[\because A_1, A_3, \ldots, A_{2n-1}$ are skew-symmetric matrices

$\therefore (A_i)^T = -A_i \ \forall \ i = 1, 3, 5, \ldots, 2n-1]$

$= -[A + 3A_3^3 + \ldots + (2n-1)A_{2n-1}^{2n-1}]$

$= -B$

Hence, B is a skew-symmetric matrix.

15 $A^T A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \times \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

$= \begin{bmatrix} a^2 + b^2 + c^2 & ab + bc + ca & ac + ab + bc \\ ab + bc + ca & b^2 + c^2 + a^2 & ab + bc + ca \\ ab + bc + ca & ab + bc + ca & a^2 + b^2 + c^2 \end{bmatrix}$

$A^T A = I \Rightarrow a^2 + b^2 + c^2 = 1$

and $\quad ab + bc + ca = 0$

Since $\quad a, b, c > 0,$

$\therefore ab + bc + ca \neq 0$ and hence no real value of $a^3 + b^3 + c^3$ exists.

16 $AA' = A'A, B = A^{-1}A'$.

$BB' = (A^{-1}A')(A^{-1} \cdot A')'$

$= (A^{-1}A')[(A')'(A^{-1})']$

$= (A^{-1}A')[A(A')^{-1}]$

$[\because (A^{-1})' = (A')^{-1}]$

$= A^{-1}(A'A)(A')^{-1}$

$= A^{-1}(AA')(A')^{-1} \quad [\because A'A = AA']$

$= (A^{-1}A)[A'(A')^{-1}]$

$= I \cdot I = I$

17 Total number of matrices $= 3^9$. A is symmetric, then $a_{ij} = a_{ji}$.

Now, 6 places (3 diagonal, 3 non-diagonal), can be filled from any of $-1, 0, 1$ in 3^6 ways. A is skew-symmetric, then diagonal entries are 'o' and a_{12}, a_{13}, a_{23} can be filled from any of $-1, 0, 1$ in 3^3 ways. Zero matrix is common.

$\therefore$ Favourable matrices are $3^9 - 3^6 - 3^3 + 1$.

Hence, required probability

$= \frac{3^9 - 3^6 - 3^3 + 1}{3^9}$